MATH. SCAND. 50 (1982), 44–54

NON-STANDARD CHARACTERIZATIONS OF IDEALS IN $C(X)$

JEPPE CHRISTOFFER DYRE

The purpose of this paper is to show how non-standard methods can be applied to the theory of $C(X)$. Here $C(X)$ is the ring of all realvalued continuous functions on a completely regular Hausdorff topological space X. The primary reference to the theory of $C(X)$ is the book by Gillman and Jerison [3]. In this book you can find the motivation for only considering completely regular spaces.

In section 1 we give a brief introduction to non-standard analysis. In the sections 2, 3, and 4 we give simple and intuitive non-standard characterizations of maximal ideals, prime ideals, z-ideals, etc. These characterizations lead to the definition of a new class of ideals, the "local" ideals, which are studied in section 5. Finally, in section 6, we prove that X is an F-space if and only if every ideal in $C(X)$ is local.

1. Non-standard analysis for pedestrians.

Non-standard analysis is an application of mathematical logic (the theory of ultraproducts). The subject was founded by Abraham Robinson, his original book [5] appeared in 1966. The method can be applied to any branch of mathematics, however, the reader should keep in mind, that any standard theorem proved by means of non-standard analysis can also be obtained by standard methods. Below we shall present the two basic features of non-standard analysis: the transfer principle and the concurrence principle.

In non-standard analysis to any set A corresponds the "*-transformed set" $^*A$. The **-transformed* of a mathematical statement is by definition the same statement with every non-quantized object replaced by its *-transformed object. By a "mathematical statement" we here mean a "formula" in the strict sense of mathematical logic (compare [1] and the examples below). The *transfer principle* tells us that a statement is true iff the *-transformed statement is true. Examples: "$\mathbf{R}$ is a field" implies that "$^*\mathbf{R}$ is a field". Or

Received August 1, 1980.

$$\forall n \in N \; \exists k \in N: \; n^n < k \quad \text{is equivalent to}$$

$$\forall n \in {}^*N \; \exists k \in {}^*N: \; n^n < k .$$

Strictly speaking we should have written $^*<$, but because of the transfer principle, $^*<$ is an ordering of $^*N$ and therefore the star is usually left out.

Some further remarks on the *-transformation: If $x \in A$ we get $^*x \in {}^*A$. Since $x \neq y$, iff $^*x \neq {}^*y$, we have a canonical inclusion $A \hookrightarrow {}^*A$ for any set A. It can be proved that $A = {}^*A$, iff A is finite [1].—An easy application of the transfer principle shows that whenever $f: A \to B$, $^*f$ is a mapping: $^*f: {}^*A \curvearrowright {}^*B$.

The transfer principle tells us something about the *-transformed of an object. Further information is gained from the *concurrence principle*: A relation $r \subseteq A \times B$ is said to be *concurrent* if

$$\forall a_1, \ldots, a_n \in A \; \exists b \in B \; \forall i \in \{1, \ldots, n\}: \; (a_i, b) \in r .$$

The concurrence principle states that for any concurrent relation $r \subseteq A \times B$ there exist $b \in {}^*B$ such that

$$\forall a \in A: \; ({}^*a, b) \in {}^*r .$$

($^*r \subseteq {}^*(A \times B) = {}^*A \times {}^*B$ by the transfer principle.) It should be noted, that this $b \in {}^*B$ is by no means uniquely determined. This is a typical feature of non-standard analysis. As an example of the concurrence principle note that $<$ on R is concurrent. Therefore there exist $M \in {}^*R$ such that $x < M$ for any $x \in R$, that is M is infinitely great.—In the practical applications of non-standard analysis the concurrence principle often replaces Zorns lemma.

We define the finite numbers F by

$$F = \{x \in {}^*R \mid \exists y \in R_+ : \; |x| \leqq y\} .$$

Obviously $R \subseteq F \subseteq {}^*R$ and F is a subring of $^*R$. For any numbers $x, y \in {}^*R$ we write $x \approx y$ ("x is infinitely close to y") whenever

$$\forall n \in N: \; |x - y| < 1/n .$$

Any number $x \in F$ lies infinitely close to a (unique) real number denoted by st (x), st: $F \curvearrowright R$ is clearly a homomorphism.

I hope the above remarks have given you a feeling of what non-standard analysis is. For further studying of the subject I warmly recommend [1].

2. The Stone–Cech compactification.

By $C_b(X)$ we understand the ring of all bounded continuous real functions on the completely regular space X.

THEOREM (2.1): *For any* $x \in {}^*X$, $\{f \in C_b(X) \mid {}^*f(x) \approx 0\}$ *is a maximal ideal in* $C_b(X)$ *and any maximal ideal is of this form.*

PROOF. The mapping $f \mapsto \mathrm{st}({}^*f(x))$ is a surjective homomorphism: $C_b(X) \curvearrowright R$ for any $x \in {}^*X$. From this it follows that the kernel $\{f \in C_b(X) \mid {}^*f(x) \approx 0\}$ is a maximal ideal. Suppose M is maximal. If $f_1, \ldots, f_n \in M$ and $\varepsilon_1, \ldots, \varepsilon_n \in \mathrm{R}_+$ there exist $x \in X$ such that

$$\forall i \in \{1, \ldots, n\} : \ |f_i(x)| < \varepsilon_i$$

since otherwise $f_1^2 + \ldots + f_n^2$ is invertible. By the concurrence principle we obtain an element $x \in {}^*X$ such that

$$\forall f \in M \ \forall \varepsilon \in R_+ : \ |{}^*f(x)| < \varepsilon, \quad \text{that is}$$

$$M \subseteqq \{f \in C_b(X) \mid {}^*f(x) \approx 0\} .$$

By the maximality of M we have equality.

The Stone–Cech compactification βX of X is the set of maximal ideals of $C_b(X)$. By (2.1) we have $\beta X = {}^*X/\underset{\mathrm{sc}}{\approx}$ where $\underset{\mathrm{sc}}{\approx}$ is defined by

$$\forall x, y \in {}^*X : \ x \underset{\mathrm{sc}}{\approx} y \Leftrightarrow [\forall f \in C_b(X) : \ {}^*f(x) \approx {}^*f(y)] .$$

It turns out to be advantageous to regard the points of βX as subsets of ${}^*X$ (compare 3.4, 3.5, and 6.1).

Let us just sketch how the topology on βX may be constructed (compare [4]): For $f \in C_b(X)$ we define $f^\beta : \beta X \to R$ by

$$f^\beta(p) = \mathrm{st}({}^*f(x)), \quad x \in p .$$

The topology on βX is the initial topology for the mappings f^β, $f \in C_b(X)$. The Hausdorff property of βX is obvious. X is canonically a subset of βX, by complete regularity X is topologically a subspace of βX. By the transfer principle X is dense in βX. Finally, a simple application of the concurrence principle shows that βX is compact.

The relevance of βX to the study of $C(X)$ is shown by the fundamental theorem of Gelfand and Kolmogoroff [3], which states that the maximal ideals of $C(X)$ are exactly the ideals M^p, $p \in \beta X$, where

$$M^p = \{f \in C(X) \mid p \in \overline{Z(f)}^\beta\} .$$

Here $\overline{Z(f)}^\beta$ is the closure of $Z(f) = f^{-1}(\{0\})$ in βX. In the next section we shall obtain a non-standard characterization of M^p.

3. Prime ideals.

In the standard theory of $C(X)$ it is difficult to construct examples of prime ideals, that are not maximal. To illustrate the usefullness of non-standard methods we start our investigation of prime ideals by giving an explicit example of a chain of prime ideals.

EXAMPLE (3.1): Put $X=[0,1]$. For $J \in {}^*R_+$ infinite we define

$$m(J) = \{x \in F \mid \exists a \in R_+ : \ x \cdot J^a \in F\} .$$

It is easy to see that $m(J)$ is a prime ideal in F. Choose $d \in {}^*R_+$ infinitely small. For any $x \in R_+$, d^{-x} and thereby $\exp(d^{-x})$ is infinite and we define

$$P(x) = \{f \in C([0,1]) \mid {}^*f(d) \in m(\exp(d^{-x}))\} .$$

If $f \in C([0,1])$, ${}^*f(d) \in F$ by the continuity of f [1] and therefore $P(x)$ is a prime ideal in $C([0,1])$. Suppose $x, y \in R_+$ and $x<y$. Then $\exp(d^{-x}) < \exp(d^{-y})$. From this we conclude that $P(y) \subseteqq P(x)$. In fact $P(y) \subset P(x)$: Define $f \in C([0,1])$ by $f(t) = \exp(-t^{-x})$, ($f(0)=0$). Obviously $f \in P(x)$. For any $a \in R_+$

$$\begin{aligned} {}^*f(d) \cdot (\exp(d^{-y}))^a &= \exp(-d^{-x} + a \cdot d^{-y}) \\ &= \exp(d^{-x}(a \cdot d^{x-y} - 1)) , \end{aligned}$$

which is an infinite number. Therefore $f \notin P(y)$. Our conclusion is that $\{P(x) \mid x \in R_+\}$ is a chain of prime ideals of cardinality $\aleph$ in $C([0,1])$. Since $\operatorname{card}(C([0,1])) = \aleph$ it is impossible to construct larger chains.

We now return to the general theory of $C(X)$.

THEOREM (3.2): *If P is a minimal prime ideal in $C(X)$, there exist $x \in {}^*X$ such that*

$$P = \{f \in C(X) \mid {}^*f(x)=0\} .$$

PROOF: Suppose $f_1, \ldots, f_n \in C(X) \setminus P$. Since P is prime $f_1 \cdot \ldots \cdot f_n \neq 0$, that is

$$\exists y \in X: \ f_1(y) \neq 0 \wedge \ldots \wedge f_n(y) \neq 0 .$$

By the concurrence principle there exist $x \in {}^*X$ obeying

$$\forall f \in C(X) \setminus P: \ {}^*f(x) \neq 0 \quad \text{or}$$

$$\{f \in C(X) \mid {}^*f(x)=0\} \subseteqq P .$$

However the left hand side is a prime ideal. By the minimality of P we must have equality.

For any $x \in {}^*X$ we define

$$P_x = \{f \in C(X) \mid {}^*f(x)=0\} .$$

Obviously P_x is a prime ideal. We now want to characterize the ideals of the form P_x. Remember that an ideal I is called a *z-ideal* [3] if

$$\forall f \in C(X)\ \forall g \in I:\ Z(g) \subseteq Z(f) \Rightarrow \quad f \in I .$$

THEOREM (3.3): *For any $x \in {}^*X$, P_x is a prime z-ideal and any prime z-ideal is of this form.*

PROOF: If $f, g \in C(X)$ and $Z(g) \subseteq Z(f)$, then $Z({}^*g) \subseteq Z({}^*f)$ by the transfer principle. From this we conclude that P_x is a prime z-ideal for any $x \in {}^*X$. Now suppose conversely that P is a prime z-ideal in $C(X)$. If $f_1, \ldots, f_n \in C(X)$ are ordered such that $f_1, \ldots, f_k \in P$ and $f_{k+1}, \ldots, f_n \in C(X) \setminus P$, we put

$$f = f_1^2 + \ldots + f_k^2 \quad \text{and} \quad g = f_{k+1} \cdot \ldots \cdot f_n .$$

Since P is prime $f \in P$ and $g \notin P$. Since P is a z-ideal we conclude that

$$\exists x \in X:\ x \in Z(f) \wedge x \notin Z(g) \quad \text{or}$$

$$\exists x \in X:\ f_1(x) = \ldots = f_k(x) = 0 \wedge f_{k+1}(x) \neq 0 \wedge \ldots \wedge f_n(x) \neq 0 .$$

The case $k=0$ is easily handled and in the case $k=n$ we note that $f \in P$ implies that f is not invertible, that is $Z(f) \neq \emptyset$. Now, by the concurrence principle, there exist $x \in {}^*X$ such that

$$\forall f \in C(X):\ {}^*f(x) = 0 \Leftrightarrow f \in P$$

or $P = P_x$.

In particular 3.2 and 3.3 tell us that any minimal prime ideal is a prime z-ideal [3]. Note that none of the prime ideals $P(x)$ in example (3.1) are z-ideals, simply because $Z(f_x) = \{0\}$ if $f_x(t) = \exp(-t^{-x})$.

Remember that for $p \in \beta X$, $M^p = \{f \in C(X) \mid p \in \overline{Z(f)}^{\beta}$.

LEMMA (3.4): *For any $x \in {}^*X$, $P_x \subseteq M^p$ implies that $x \in p$.*

PROOF: Suppose that $x \in q$, $q \neq p$. Choose h_1, h_2 and f in $C_b(X)$ with the following properties:

$$q \in (h_1^\beta)^{-1}(]-1,1[), \qquad p \in (h_2^\beta)^{-1}(]-1,1[)$$

$$f^\beta((h_1^\beta)^{-1}(]-1,1[)) = \{0\}$$

$$f^\beta((h_2^\beta)^{-1}(]-1,1[)) = \{1\} .$$

Obviously $f \notin M^p$ and

$$\forall y \in X: \ |h_1(y)| < 1 \Rightarrow f(y) = 0 .$$

By the transfer principle

$$\forall y \in {}^*X: \ |{}^*h_1(y)| < 1 \Rightarrow {}^*f(y) = 0 .$$

In particular ${}^*f(x)=0$. This contradicts the assumption $P_x \subseteqq M^p$.

Since M^p is itself a prime z-ideal, M^p is union of the prime z-ideals contained in M^p. By 3.3 and 3.4 this gives us the following non-standard representation of M^p:

$$M^p = \{f \in C(X) \mid \exists x \in p: \ {}^*f(x)=0\} .$$

If we define O^p as the intersection of the prime ideals contained in M^p, we can write

$$O^p = \{f \in C(X) \mid \forall x \in p: \ {}^*f(x)=0\}$$

since any prime ideal contains a minimal prime ideal by Zorns lemma (compare 3.2). From the above results it is easy to prove the important theorem by Gillman, Jerison, and Henriksen [3] that for any prime ideal P in $C(X)$ there exist a unique point $p \in \beta X$ such that

$$O^p \subseteqq P \subseteqq M^p .$$

The standard [3] and non-standard representations of M^p and O^p can be summarized to

THEOREM (3.5): *For $p \in \beta X$*

$$M^p = \{f \in C(X) \mid p \in \overline{Z(f)}^{\beta}\} = \{f \in C(X) \mid \exists x \in p: {}^*f(x)=0\}$$

$$O^p = \{f \in C(X) \mid p \in \overline{\overset{\circ}{Z(f)}}^{\beta}\} = \{f \in C(X) \mid \forall x \in p: {}^*f(x)=0\} .$$

From this we obtain

COROLLARY (3.6): *For any $f \in C(X)$*

$$\overline{Z(f)}^{\beta} = \{p \in \beta X \mid p \cap Z({}^*f) \neq \emptyset\} ,$$

$$\overline{\overset{\circ}{Z(f)}}^{\beta} = \{p \in \beta X \mid p \subseteqq Z({}^*f)\} .$$

For any $x \in {}^*X$ we define R_x and i_x: $C(X) \curvearrowright R_x$ by

$$R_x = \{{}^*f(x) \in {}^*R \mid f \in C(X)\}, \quad i_x(f) = {}^*f(x) .$$

R_x is of course $C(X)/P_x$ and i_x is the quotient mapping: $C(X) \curvearrowright C(X)/P_x$.

THEOREM (3.7): *An ideal P in $C(X)$ is a prime ideal if and only if there exist $x \in {}^*X$ and a prime ideal Q in R_x such that*

$$P = \{f \in C(X) \mid {}^*f(x) \in Q\} .$$

PROOF. Given P prime. Choose by 3.2, $x \in {}^*X$ such that $P_x \subseteqq P$. Put $Q = i_x(P)$. Q is a prime ideal in R_x and

$$P = \{f \in C(X) \mid {}^*f(x) \in Q\} .$$

The converse is trivial.

The above theorem tells us that the property of being a prime ideal is locally determined. I think that (3.7) gives a picture of how prime ideals look.

4. Some other types of ideals.

The non-standard characterization of z-ideals is the following:

THEOREM (4.1): *An ideal I is a z-ideal if and only if there exist a nonempty set $A \subseteqq {}^*X$ such that*

$$I = \{f \in C(X) \mid A \subseteqq Z({}^*f)\} .$$

The proof of 4.1 proceeds in a manner similar to the proof of 3.3.

Comparing 4.1 with 3.3, we get the well-known result [3] that any z-ideal is an intersection of prime z-ideals, and also that the set A can be chosen as a one-point set iff I is prime. Note also that O^p is a z-ideal, and that in this case the set A can be chosen as the "point" $p \in \beta X$ (3.5).

We recall that an ideal I is called *pseudoprime* [2] if I contains a prime ideal. Analogous to 3.7 it is easy to prove the following theorem, which gives us a non-standard characterization of pseudoprime ideals:

THEOREM (4.2): *An ideal I is pseudoprime if and only if there exist $x \in {}^*X$ and an ideal $A_x \subseteqq R_x$ such that*

$$I = \{f \in C(X) \mid {}^*f(x) \in A_x\} .$$

EXAMPLE: An ideal which is pseudoprime but not prime: Choose $d \in {}^*R_+$ infi[illegible] Put

$$[illegible] \; {}^*f(d)/d \in F\} .$$

I is easily seen to be an ideal and since $P_d \subseteqq I$, I is pseudoprime. The function $|x|$ belongs to I. If I was prime then $\sqrt{|x|} \in I$ but $\sqrt{d}/d = (\sqrt{d})^{-1} \notin F$.

Finally, let us consider absolutely convex ideals:
An ideal I is called *absolutely convex* [3] if

$$\forall f \in C(X)\ \forall g \in I:\ |f| \leqq |g| \Rightarrow f \in I .$$

THEOREM (4.3): *An ideal I is absolutely convex if and only if there exist a mapping a: $^*X \curvearrowright {}^*R_+$ such that*

$$I = \{f \in C(X) \mid \forall x \in {}^*X: |{}^*f(x)| < a(x)\} .$$

The proof is left to the reader.

5. Local ideals.

We have seen that z-ideals, pseudoprime ideals and absolutely convex ideals all somehow are locally determined (4.1, 4.2, and 4.3). This section contains a complete description of locally determined ideals (5.2 and 5.3).

DEFINITION: An ideal I is called *local* if

$$\forall f \in C(X)\ \forall f_1, \ldots, f_n \in I:\ (f-f_1)\cdot \ldots \cdot (f-f_n) = 0 \Rightarrow f \in I .$$

THEOREM (5.1): *Any z-ideal, any pseudoprime ideal and any absolutely convex ideal is local.*

PROOF: Suppose I is an ideal, $f \in C(X), f_1, \ldots, f_n \in I$, and $(f-f_1)\cdot \ldots \cdot (f-f_n) = 0$.

1) I is pseudoprime: By 4.2 and the transfer principle we conclude that there exist $i \in \{1, \ldots, n\}$ such that $f - f_i \in I$. This implies that $f \in I$.

2) I is absolutely convex: $|f_1| + \ldots + |f_n| \in I$, and from $|f| \leqq |f_1| + \ldots + |f_n|$ we conclude that $f \in I$. Since any z-ideal is absolutely convex the proof is finished.

The following two theorems justifies the name "local".

THEOREM (5.2): *If I is an ideal which can be written*

$$I = \bigcap_{x \in B} \{f \in C(X) \mid {}^*f(x) \in A_x\}$$

for suitable sets $B \subseteqq {}^*X$, $(B \neq \emptyset)$ and $A_x \subseteqq {}^*R$, I is local.

PROOF: Suppose $f_1,\ldots,f_n \in I$, $f \in C(X)$ and $(f-f_1)\cdot\ldots\cdot(f-f_n)=0$. Then by the transfer principle

$$\forall x \in {}^*X \; \exists i \in \{1,\ldots,n\}: \; {}^*f(x) = {}^*f_i(x) .$$

But this implies that $f \in I$.

Note that 5.2 in conjunction with 4.1, 4.2, and 4.3 in fact proves 5.1.

THEOREM (5.3): *An ideal I is local if and only if*

$$I = \{f \in C(X) \mid \forall x \in {}^*X: \; {}^*f(x) \in i_x(I)\} .$$

PROOF: $\Leftarrow$ follows from 5.2. $\Rightarrow$ The inclusion $\subseteq$ is valid for any ideal. If $f \in C(X) \setminus I$ and $f_1,\ldots,f_n \in I$ the locality of I implies that

$$\exists x \in X: \; f(x) \neq f_1(x) \wedge \ldots \wedge f(x) \neq f_n(x) .$$

The concurrence principle now tells us that there exist a point $x \in {}^*X$ such that

$$\forall g \in I: \; {}^*f(x) \neq {}^*g(x) .$$

From this we conclude that f does not belong to the right hand side of 5.3.

THEOREM (5.4): *An ideal I is local if and only if I is an intersection of pseudoprime ideals.*

PROOF. By 5.3 and 4.2 any local ideal is an intersection of pseudoprime ideals. On the other hand any intersection of local ideals clearly is local. By 5.1 we then get the converse.

Theorem 5.4 should be compared with the well-known result in algebra, that an ideal I in a commutative ring R is an intersection of prime ideals iff R/I contains no nontrivial nilpotent elements.

The intersection of all local ideals that contains a given ideal I is denoted by $\bar{I}^L$ ("the *local closure of* I"). $\bar{I}^L$ is the smallest local ideal that contains I. Of course I is local iff $I = \bar{I}^L$. The next two theorems give us non-standard and standard characterizations of $\bar{I}^L$ for an ideal I.

THEOREM (5.5): *For any ideal I*

$$\bar{I}^L = \{f \in C(X) \mid \forall x \in {}^*X: \; {}^*f(x) \in i_x(I)\} .$$

PROOF: The right hand side is a local ideal containing I. If H is a local ideal and $I \subseteqq H$ we have by 5.3

$$\{f \in C(X) \mid \forall x \in {}^*X: {}^*f(x) \in i_x(I)\}$$
$$\subseteqq \{f \in C(X) \mid \forall x \in {}^*X: {}^*f(x) \in i_x(H)\} = H .$$

THEOREM (5.6): *For any ideal I,*

$$\bar{I}^L = \{f \in C(X) \mid \exists f_1, \ldots, f_n \in I: (f-f_1) \cdot \ldots \cdot (f-f_n) = 0\} .$$

PROOF: It is easy to see that the right hand side is a local ideal containing I. On the other hand any local ideal containing I clearly contains the right hand side.

EXAMPLE: Put $X = R$ and consider the principal ideal generated by the identical function (x). Since $(|x|+x)\cdot(|x|-x)=0$ and $|x| \notin (x)$, (x) is not local. We leave it to the reader to verify that

$$\overline{(x)}^L = \{f \in C(R) \mid \lim_{x \to 0, x>0} f(x)/x \quad \text{and} \quad \lim_{x \to 0, x<0} f(x)/x \text{ exist}\} .$$

If we however restrict ourselves to consider the topological space $X = [0, \infty[$ we shall find that (x) *is* local. This is an example of a local ideal that is not absolutely convex ($|x \cdot \sin(x^{-1})| \leqq |x|$).

6. F-spaces and local ideals.

Recall the following definitions [3]: A point $p \in \beta X$ is called an *F-point* if O^p is prime. X is called an *F-space* if all points of βX are F-points.

THEOREM (6.1): *For a point $p \in \beta X$ the following conditions are equivalent*:

1) *p is an F-point,*
2) $\exists x \in p \; \forall f \in C(X)$: ${}^*f(x) = 0 \Rightarrow [\forall y \in p$: ${}^*f(y)=0]$,
3) $\forall f \in C(X)$: $(\forall y \in p$: ${}^*f(y) \geqq 0) \vee (\forall y \in p$: ${}^*f(y) \leqq 0)$.

PROOF: 1) $\Leftrightarrow$ 2): Since O^p is a z-ideal, it is prime iff it is a prime z-ideal. By 3.3 we obtain the wanted equivalence.

2) $\Rightarrow$ 3): Choose x according to 2). If $f \in C(X)$ and ${}^*f(x) \geqq 0$, then ${}^*(\min\{f, 0\})(x) = 0$. By 2) we get ${}^*f(y) \geqq 0$ for any $y \in p$. Similarly ${}^*f(x) \leqq 0$ implies $\forall y \in p$: ${}^*f(y) \leqq 0$.

3) $\Rightarrow$ 1): Suppose $f, g \in C(X)$ and $f \cdot g \in O^p$. If $(|{}^*f| - |{}^*g|)(y) \geqq 0$ for every $y \in p$ we must have $\forall y \in p$: ${}^*g(y) = 0$ since otherwise there would exist $y_0 \in p$

such that $*(fg)(y_0) \neq 0$ (compare 3.5). Correspondingly $\forall y \in p$: $(|*f| - |*g|)(y) \leq 0$ implies $f \in O^p$.

Theorem 14.25 of [3] shows that there exist many characterizations of F-spaces by means of the algebraic structure of $C(X)$. For instance X is an F-space iff every finitely generated ideal in $C(X)$ is principal. In [2] Gillman and Kohls gave another characterization of F-spaces: X is an F-space iff every ideal in $C(X)$ is an intersection of pseudoprime ideals. By 5.4 we therefore have:

THEOREM (6.2). *X is an F-space if and only if every ideal in $C(X)$ is local.*

It is possible to prove $\Rightarrow$ in 6.2 by means of 3.6, 6.1, and a suitable finite partition of unity on βX (compare [2]). $\Leftarrow$ in 6.2 can be proved by combining 6.1 with the fact that for any $f \in C(X)$ the principal ideal (f) contains $|f|$. In particular we notice that 6.2 can be strengthened to: X is an F-space iff every principal ideal in $C(X)$ is local.

I would like to thank my teacher Anton Jensen and my friends Jens Corfitzen and Lars Gerhard Jensen for stimulating conversations.

REFERENCES

1. M. Davis, *Applied nonstandard analysis*, John Wiley & Sons, New York, London, Sydney, Toronto, 1977.
2. L. Gillman and C. W. Kohls, *Convex and pseudoprime ideals in rings of continuous functions*, Math. Z. 72 (1960), 399–409.
3. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, Toronto, London, 1960.
4. W. A. J. Luxemburg and K. D. Stroyan, *Introduction to the theory of infinitesimals* (Pure and Applied Mathematics 72), Academic Press, New York, San Francisco, London, 1976.
5. A. Robinson, *Non-standard analysis*, North-Holland Publishing Company, Amsterdam, 1966.

MATEMATISK INSTITUT
H. C. ØRSTED INSTITUTTET
UNIVERSITETSPARKEN 5
2100 KØBENHAVN Ø
DENMARK